

Applications of Information Nonanticipative Rate Distortion Function

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Abstract—The objective of this paper is to further investigate various applications of information Nonanticipative Rate Distortion Function (NRDF) by discussing two working examples, the Binary Symmetric Markov Source with parameter p (BSMS(p)) with Hamming distance distortion, and the multidimensional partially observed Gaussian-Markov source. For the BSMS(p), we give the solution to the NRDF, and we use it to compute the Rate Loss (RL) of causal codes with respect to noncausal codes. For the multidimensional Gaussian-Markov source, we give the solution to the NRDF, we show its operational meaning via joint source-channel matching over a vector of parallel Gaussian channels, and we compute the RL of causal and zero-delay codes with respect to noncausal codes.

I. INTRODUCTION

In this paper, we consider an information theoretic measure called Nonanticipative Rate Distortion Function (NRDF) [1], [2] which is a variation of the classical RDF [3], and we discuss some of its applications in problems on information theory. In [1], it is pointed out that the information NRDF and nonanticipatory ϵ -entropy introduced in [4] to facilitate real-time applications are equivalent notions, and a variational equality is derived and utilized to introduce a Blahut-Arimoto Algorithm (BAA) to iteratively compute the information NRDF. In addition, existence of the optimal nonanticipative reproduction conditional distribution is shown, under the topology of weak convergence of probability measures, while in [2], the closed form expression of the optimal reproduction conditional distribution for stationary processes is derived. Moreover, in [2], the realization of the optimal reproduction distribution of the information NRDF is discussed (see Fig. 3) in the context of filtering applications with fidelity constraints. In this paper, we present results in the following directions.

(R1) Compute the NRDF in closed form for two examples of sources with memory: (a) the *Binary Symmetric Markov Source with parameter p with Hamming distortion* (BSMS(p)), for which the classical RDF is only known in the distortion region $0 \leq D \leq D_c$ [5], while for the rest $D_c \leq D \leq \frac{1}{2}$ only upper and lower bounds are known [6]. We show that the solution of the NRDF is a tight upper bound for $D_c \leq D \leq \frac{1}{2}$, and performs much more reliably in comparison to the upper bound found in [6]; (b) the *multidimensional Gaussian-Markov source*, for which only upper bounds are known, since no closed form expression is given in the literature apart from the first-order (scalar) Gauss-Markov sources [7, Th. 3].

(R2) Compute the Rate Loss (RL) of causal codes, that

is, the gap between the Optimal Performance Theoretically Attainable (OPTA) by causal codes with respect to noncausal codes for the BSMS(p).

(R3) Compute the RL of causal and zero-delay codes with respect to noncausal codes for the multidimensional Gaussian-Markov source, and show achievability of the NRDF using symbol-by-symbol transmission [8].

(R4) Provide an alternative characterization of the closed form expression to the information NRDF, from which a lower bound on the NRDF similar to Shannon's Lower Bound (SLB) [3, Ch. 4] can be derived, for any source with memory, including Gaussian-Markov sources. This bound is utilized in the derivation of the closed form expression of the multidimensional Gaussian-Markov source.

The alternative characterization of the solution to the information NRDF (see Theorem 3) is the analogue of the single letter characterization of the classical RDF of discrete memoryless sources, often used to facilitate the computation of the classical BAA [9, Th. 6.3.9].

Finally, we point out that the multidimensional Gaussian-Markov source example is a generalization to arbitrary dimensions of the example considered in [10, Cor. 1.2] for systems with low delay tolerance at both the encoder and decoder, such as, the classical Differential Predictive Coded Modulation (DPCM) system [3], often applied to compression applications of video, audio, image, and speech coding.

II. NRDF ON ABSTRACT SPACES

In this section, we define the information NRDF by adopting the general mathematical framework described in [1].

Notation. Let $\mathbb{N} \triangleq \{0, 1, \dots\}$. Introduce two sequence of spaces $\{(\mathcal{X}_n, \mathcal{B}(\mathcal{X}_n)) : n \in \mathbb{N}\}$ and $\{(\mathcal{Y}_n, \mathcal{B}(\mathcal{Y}_n)) : n \in \mathbb{N}\}$, where $\mathcal{X}_n, \mathcal{Y}_n, n \in \mathbb{N}$, are Polish spaces, and $\mathcal{B}(\mathcal{X}_n)$ and $\mathcal{B}(\mathcal{Y}_n)$ are Borel σ -algebras of subsets of \mathcal{X}_n and \mathcal{Y}_n , respectively. Points in $\mathcal{X}^{\mathbb{N}} \triangleq \times_{n \in \mathbb{N}} \mathcal{X}_n$ are denoted by $\mathbf{x} \triangleq \{x_0, x_1, \dots\} \in \mathcal{X}^{\mathbb{N}}$, while their restrictions to finite coordinates are denoted by $x^n \triangleq \{x_0, x_1, \dots, x_n\} \in \mathcal{X}_{0,n}$, for $n \in \mathbb{N}$, and similarly of \mathcal{Y}_n . Let $\mathcal{B}(\mathcal{X}^{\mathbb{N}}) \triangleq \odot_{i \in \mathbb{N}} \mathcal{B}(\mathcal{X}_i)$ denote the σ -algebra on $\mathcal{X}^{\mathbb{N}}$ generated by cylinder sets and similarly for $\mathcal{B}(\mathcal{Y}^{\mathbb{N}}) \triangleq \odot_{i \in \mathbb{N}} \mathcal{B}(\mathcal{Y}_i)$, while $\mathcal{B}(\mathcal{X}_{0,n})$ and $\mathcal{B}(\mathcal{Y}_{0,n})$ denote the σ -algebras with bases over $A_i \in \mathcal{B}(\mathcal{X}_i)$, and $B_i \in \mathcal{B}(\mathcal{Y}_i)$, $i = 0, 1, \dots, n$, respectively. Let $\mathcal{Q}(\mathcal{Y}; \mathcal{X})$ denote the set of stochastic kernels on \mathcal{Y} given \mathcal{X} and $\mathcal{M}(\mathcal{X})$

the set of probability measures on \mathcal{X} .

Source Distribution. Consider the sequence of source distributions $\{p_n(dx_n; x^{n-1}) : n \in \mathbb{N}\}$, where $p_n(\cdot; \cdot) \in \mathcal{Q}(\mathcal{X}_n; \mathcal{X}_{0,n-1})$. For $A \in \mathcal{B}(\mathcal{X}_{0,n})$ a cylinder set of the form $A \triangleq \{\mathbf{x} \in \mathcal{X}^{\mathbb{N}} : x_0 \in A_0, x_1 \in A_1, \dots, x_n \in A_n\}$, $A_i \in \mathcal{B}(\mathcal{X}_i)$, $i = 0, 1, \dots, n$, we define $\mathbf{P}(\cdot)$ on $\mathcal{B}(\mathcal{X}^{\mathbb{N}})$ by

$$\mathbf{P}(A) \triangleq \int_{A_0} p_0(dx_0) \dots \int_{A_n} p_n(dx_n; x^{n-1}) \equiv \mu_{0,n}(A_{0,n}) \quad (1)$$

where $A_{0,n} = \times_{i=0}^n A_i$, and $\mu_{0,n}(\cdot)$ denotes the restriction of the measure $\mathbf{P}(\cdot)$ on cylinder sets $A \in \mathcal{B}(\mathcal{X}_{0,n})$, for $n \in \mathbb{N}$.

Reproduction Distribution. Consider the sequence of reproduction distributions $\{q_n(dy_n; y^{n-1}, x^n) : n \in \mathbb{N}\}$, where $q_n(\cdot; \cdot, \cdot) \in \mathcal{Q}(\mathcal{Y}_n; \mathcal{Y}_{0,n-1} \times \mathcal{X}_{0,n})$. For a cylinder set $B \triangleq \{\mathbf{y} \in \mathcal{Y}^{\mathbb{N}} : y_0 \in B_0, y_1 \in B_1, \dots, y_n \in B_n\}$, we define $\mathbf{Q}(\cdot|\mathbf{x})$ on $\mathcal{B}(\mathcal{Y}^{\mathbb{N}})$ by

$$\mathbf{Q}(B|\mathbf{x}) \triangleq \int_{B_0} q_0(dy_0; x_0) \dots \int_{B_n} q_n(dy_n; y^{n-1}, x^n) \quad (2)$$

$$\equiv \vec{Q}_{0,n}(B_{0,n}|x^n), \quad B_{0,n} \in \mathcal{B}(\mathcal{Y}_{0,n}). \quad (3)$$

For Polish spaces, it can be shown [11, Sec. II] that any family of measures $\mathbf{Q}(\cdot|\mathbf{x})$ on $\mathcal{B}(\mathcal{Y}^{\mathbb{N}})$ defined by (2) is equivalent to a family of measures $\mathbf{Q}(\cdot|\mathbf{x})$ on $(\mathcal{Y}^{\mathbb{N}}, \mathcal{B}(\mathcal{Y}^{\mathbb{N}}))$ satisfying the following consistency condition.

C1: If $D \in \mathcal{B}(\mathcal{Y}_{0,n})$, then $\mathbf{Q}(D|\mathbf{x})$ is $\mathcal{B}(\mathcal{X}_{0,n})$ -measurable function of $\mathbf{x} \in \mathcal{X}^{\mathbb{N}}$.

We denote the set of measures satisfying **C1** by $\mathcal{Q}^{\mathbf{C1}}(\mathcal{Y}^{\mathbb{N}}; \mathcal{X}^{\mathbb{N}}) \subseteq \mathcal{Q}(\mathcal{Y}^{\mathbb{N}}; \mathcal{X}^{\mathbb{N}})$.

Indeed, for any family of measures $\mathbf{Q}(\cdot|\mathbf{x})$ on $(\mathcal{Y}^{\mathbb{N}}, \mathcal{B}(\mathcal{Y}^{\mathbb{N}}))$ satisfying consistency condition **C1** one can construct a collection of probability distributions $\{q_n(dy_n; y^{n-1}, x^n) : n \in \mathbb{N}\}$ which are connected to $\mathbf{Q}(\cdot|\mathbf{x})$ via relation (2) [11, Sec. II]. Here, $\vec{Q}_{0,n}(\cdot|x^n) \in \mathcal{Q}^{\mathbf{C1}}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n})$ denotes the restriction of $\mathbf{Q}(\cdot|\mathbf{x}) \in \mathcal{Q}^{\mathbf{C1}}(\mathcal{Y}^{\mathbb{N}}; \mathcal{X}^{\mathbb{N}})$ to finite coordinates.

Next, we introduce the precise definition of information NRDF by using relative entropy. Given $\mathbf{P}(\cdot) \in \mathcal{M}(\mathcal{X}^{\mathbb{N}})$ and $\mathbf{Q}(\cdot|\cdot) \in \mathcal{Q}^{\mathbf{C1}}(\mathcal{Y}^{\mathbb{N}}; \mathcal{X}^{\mathbb{N}})$ we define the joint distribution on $\mathcal{X}^{\mathbb{N}} \times \mathcal{Y}^{\mathbb{N}}$ by $P_{0,n}(dx^n, dy^n) \triangleq (\mu_{0,n} \otimes \vec{Q}_{0,n})(dx^n, dy^n)$, the marginal distribution on $\mathcal{Y}^{\mathbb{N}}$ by $\nu_{0,n}(dy^n) \triangleq (\mu_{0,n} \otimes \vec{Q}_{0,n})(\mathcal{X}_{0,n}, dy^n)$, and the product distribution $\Pi_{0,n} : \mathcal{B}(\mathcal{X}_{0,n}) \odot \mathcal{B}(\mathcal{Y}_{0,n}) \mapsto [0, 1]$ by

$$\begin{aligned} \Pi_{0,n}(dx^n, dy^n) &\triangleq (\mu_{0,n} \times \nu_{0,n})(dx^n, dy^n) \\ &\triangleq \int_{A_0} p_0(dx_0) \dots \int_{A_n} p_n(x_n; x^{n-1}) \int_{B_n} \nu_n(dy_n; y^{n-1}). \end{aligned}$$

The information theoretic measure of interest is a special case of directed information [1, Sec. IV] defined by relative entropy $\mathbb{D}(\cdot||\cdot)$

$$I_{\mu_{0,n}}(X^n \rightarrow Y^n) \triangleq \mathbb{D}(\mu_{0,n} \otimes \vec{Q}_{0,n} || \Pi_{0,n}) \quad (4)$$

$$= \int \log \left(\frac{\vec{Q}_{0,n}(dy^n|x^n)}{\nu_{0,n}(dy^n)} \right) (\mu_{0,n} \otimes \vec{Q}_{0,n})(dx^n, dy^n) \quad (5)$$

$$\equiv \mathbb{I}_{X^n \rightarrow Y^n}(\mu_{0,n}, \vec{Q}_{0,n}). \quad (6)$$

The notation $\mathbb{I}_{X^n \rightarrow Y^n}(\cdot, \cdot)$ indicates the functional dependence of $I_{\mu_{0,n}}(X^n \rightarrow Y^n)$ on $\{\mu_{0,n}, \vec{Q}_{0,n}\}$. Consider a measurable distortion function $d_{0,n}(x^n, y^n) : \mathcal{X}_{0,n} \times \mathcal{Y}_{0,n} \mapsto [0, \infty]$, $d_{0,n} = \sum_{i=0}^n \rho(x_i, y_i)$, and define the fidelity of reproduction by

$$\mathcal{Q}_{0,n}^{\mathbf{C1}}(D) \triangleq \left\{ \vec{Q}_{0,n}(\cdot|x^n) \in \mathcal{Q}^{\mathbf{C1}}(\mathcal{Y}_{0,n}; \mathcal{X}_{0,n}) : \frac{1}{n+1} \int d_{0,n}(x^n, y^n) (\mu_{0,n} \otimes \vec{Q}_{0,n})(dx^n, dy^n) \leq D \right\}, \quad D \geq 0.$$

Next, we define the information NRDF.

Definition 1. (Information NRDF) The information NRDF is

$$R_{0,n}^{na}(D) \triangleq \inf_{\vec{Q}_{0,n}(\cdot|x^n) \in \mathcal{Q}_{0,n}^{\mathbf{C1}}(D)} \mathbb{I}_{X^n \rightarrow Y^n}(\mu_{0,n}, \vec{Q}_{0,n}). \quad (7)$$

If the infimum over $\mathcal{Q}_{0,n}^{\mathbf{C1}}(D)$ in (7) does not exist then we set $R_{0,n}^{na}(D) = \infty$. The information NRDF rate is

$$R^{na}(D) = \lim_{n \rightarrow \infty} \frac{1}{n+1} R_{0,n}^{na}(D) \quad (8)$$

provided the limit on the right hand side (RHS) of (8) exists (if not we use $\limsup_{n \rightarrow \infty}$). If the infimum over $\mathcal{Q}_{0,n}^{\mathbf{C1}}(D)$ does not exist then we set $R^{na}(D) = \infty$.

Note that $R_{0,n}^{na}(D)$ is also related to classical RDF [3], denoted by $R_{0,n}(D)$, as follows. Let $\mathcal{Q}_{0,n}(D) = \{P_{Y^n|X^n}(\cdot|x^n) : \frac{1}{n+1} \int d_{0,n}(x^n, y^n) P_{Y^n|X^n}(dy^n|x^n) \otimes P_{X^n}(dx^n) \leq D\}$, $D \geq 0$, then

$$\begin{aligned} R_{0,n}(D) &= \inf_{P_{Y^n|X^n}(\cdot|x^n) \in \mathcal{Q}_{0,n}(D)} \mathbb{D}(P_{Y^n|X^n} \otimes P_{X^n} || P_{Y^n} \times P_{X^n}) \\ &\stackrel{(a)}{\leq} \inf_{P_{Y^n|X^n}(\cdot|x^n) \in \mathcal{Q}_{0,n}(D) \cap \mathcal{Q}_{0,n}^{\mathbf{C1}}(D)} \mathbb{D}(P_{Y^n|X^n} \otimes P_{X^n} || P_{Y^n} \times P_{X^n}). \end{aligned}$$

For memoryless sources, (a) holds with equality.

III. OPTIMIZATION OF NRDF AND PROPERTIES

In this section, we state conditions for the existence of solution to the extremum problem (7), we give the optimal reproduction minimizing (7) and some of its properties. These results are used when we discuss the various applications.

The following existence result is outlined in [1], while a complete derivation is given in [12, Sec. III].

Theorem 1. [12, Sec. III](Existence) Suppose **(A1)** $\mathcal{Y}_{0,n}$ is a compact; **(A2)** for all $h(\cdot) \in BC(\mathcal{Y}_n)$, $(x^n, y^{n-1}) \in \mathcal{X}_{0,n} \times \mathcal{Y}_{0,n-1} \mapsto \int_{\mathcal{Y}_n} h(y) P_{Y|Y^{n-1}, X^n}(dy|y^{n-1}, x^n) \in \mathbb{R}$ is continuous jointly in $(x^n, y^{n-1}) \in \mathcal{X}_{0,n} \times \mathcal{Y}_{0,n-1}$; **(A3)** $d_{0,n}(x^n, \cdot)$ is continuous on $\mathcal{Y}_{0,n}$; **(A4)** There exist $(x^n, y^n) \in \mathcal{X}_{0,n} \times \mathcal{Y}_{0,n}$ such that $d_{0,n}(x^n, y^n) < D$.

Then the infimum in $R_{0,n}^{na}(D)$ is achieved by some $\vec{Q}_{0,n}^*(dy^n|x^n) \in \mathcal{Q}_{0,n}^{\mathbf{C1}}(D)$.

It can be easily shown that $R_{0,n}^{na}(D)$ is equivalent to Gorbunov and Pinsker [4] definition of nonanticipatory ϵ -entropy defined via mutual information $I(X^n; Y^n)$ by $R_{0,n}^\epsilon(D) = \inf \{I(X^n; Y^n) : P_{Y^n|X^n}(\cdot|x^n) \in \mathcal{Q}_{0,n}(D) \cap \{X_{i+1}^n \leftrightarrow X^i \leftrightarrow Y^i, i = 0, 1, \dots, n-1\}\}$. An extensive elaboration on

the equality is given in [12, Sec. III]. By combining Theorem 1 and [4, Th. 2-4] we have the following important results.

Corollary 1. *Suppose the conditions of Theorem 1 hold. In addition, assume (A5) the source is stationary; (A6) for any $k = 1, 2, \dots$, the sets $\mathcal{Q}_{0,n}^{\mathbf{C}1}(D)$ and $\mathcal{Q}_{k,n+k}^{\mathbf{C}1}(D)$ are copies of the same set.*

Then $\lim_{n \rightarrow \infty} \frac{1}{n+1} R_{0,n}^{na}(D)$ exists and it is finite.

If also, (A7) $\vec{Q}_{0,n}(\cdot|x^n) \in \mathcal{Q}_{0,n}^{\mathbf{C}1}(D)$ implies $\vec{Q}_{0,k}(\cdot|x^k) \in \mathcal{Q}_{0,k}^{\mathbf{C}1}(D)$, $\vec{Q}_{k+1,n}(\cdot|x_{k+1}, \dots, x_n) \in \mathcal{Q}_{k+1,n}^{\mathbf{C}1}(D) \forall k = 0, 1, \dots, n-1$; (A8) for any $\alpha_t : [s_1, s_2] \mapsto [0, \infty)$, $\sum_{t=s_1}^{s_2} \alpha_t = 1$, $\forall t \in (0, \infty)$, and $P_{Y^n|X^n}(\cdot|x^n) \in \mathcal{Q}_{0,n}(D) \Rightarrow P_{\tilde{Y}^n|X^n}(\cdot|x^n) \in \mathcal{Q}_{0,n}(D)$, $\forall n \in \mathbb{N}$, where $P_{X^n, \tilde{Y}^n}(\mathcal{A}) = (P_{\tilde{Y}^n|X^n} \otimes P_{X^n})(\mathcal{A}) \triangleq \sum_{s_1}^{s_2} \alpha_s (P_{\tilde{Y}^n|X^n} \otimes P_{X^n})(\mathcal{A}_s)$, $\mathcal{A} \triangleq \{(X_i, Y_i) = (x_i, y_i) : i = 0, 1, \dots\} \subseteq \mathcal{X}^{\mathbb{N}} \times \mathcal{Y}^{\mathbb{N}}$, $\mathcal{A}_s \triangleq \{(X_{i-s}, Y_{i-s}) = (x_i, y_i) : i = 0, 1, \dots\}$. Then the infimum in (7) is achieved by $\vec{Q}_{0,n}^(dy^n|x^n) \in \mathcal{Q}_{0,n}^{\mathbf{C}1}(D)$ and $\{(X_n, Y_n) : n \in \mathbb{N}\}$ is jointly stationary.*

Proof: By Theorem 1, $R_{0,n}^{na}(D)$ is finite for any finite n . Using this and [4, Theorem 2-4], the results follow. ■

Utilizing the convexity of the extremum problem (7) (see [12, Th. II.2]), and applying variational methods, the general closed form expression of the optimal stationary reproduction conditional distribution of (7) is derived in [2, Sec. IV]. Here, we only state the main theorem.

Theorem 2. [2] *(Optimal stationary reproduction distribution) We suppose the optimal reproduction distribution and source distribution are stationary, i.e., conditions of Corollary 1 hold. The optimal solution of information NRDF is given by¹*

$$\begin{aligned} \vec{Q}_{0,n}^*(dy^n|x^n) &= \otimes_{i=0}^n q_i^*(dy_i; y^{i-1}, x^i) \\ &= \otimes_{i=0}^n \frac{e^{s\rho(x_i, y_i)} \nu_i^*(dy_i; y^{i-1})}{\int_{\mathcal{Y}_i} e^{s\rho(x_i, y_i)} \nu_i^*(dy_i; y^{i-1})}, \quad s \leq 0 \end{aligned} \quad (9)$$

and $\nu_i^*(\cdot; y^{i-1}) \in \mathcal{Q}(\mathcal{Y}_i; \mathcal{Y}_{0,i-1})$. The information NRDF is given by

$$\begin{aligned} R_{0,n}^{na}(D) &= sD(n+1) - \sum_{i=0}^n \int \log \left(\int e^{s\rho(x_i, y_i)} \nu_i^*(dy_i; y^{i-1}) \right) \\ &\quad \times \vec{Q}_{0,i-1}^*(dy^{i-1}|x^{i-1}) \otimes \mu_{0,i}(dx^i). \end{aligned}$$

Moreover, if $R_{0,n}^{na}(D) > 0$ then $s < 0$, and

$$\frac{1}{n+1} \sum_{i=0}^n \int \rho(x_i, y_i) \vec{Q}_{0,i}^*(dy^i|x^i) \otimes \mu_{0,i}(dx^i) = D.$$

Remark 1. *Note that for single letter distortion function $\rho(x_i, y_i)$ the optimal reproduction is Markov with respect to x_i given by $q_i^*(dy_i; y^{i-1}, x_i)$. If the distortion function is generalized to $\rho(x_i, T^i y^n)$, where $T^i y^n$ is the shift operator on y^n , then $\vec{Q}_{0,n}^*(dy^n|x^n)$ is given by (9) with $\rho(x_i, y_i)$ replaced by $\rho(x_i, T^i y^n)$, and similarly for $\rho(T^i x^n, y_i)$.*

¹Due to stationarity assumption $\nu_i^*(\cdot; \cdot) = \nu^*(\cdot; \cdot)$ and $q_i^*(\cdot; \cdot, \cdot) = q^*(\cdot; \cdot, \cdot)$.

Next, we present an alternative equivalent characterization of the solution of $R_{0,n}^{na}(D)$, which can be used to derive a lower bound on $R_{0,n}^{na}(D)$ similar to the SLB [3, Ch. 4].

Theorem 3. *(Alternative characterization) Suppose the conditions of Theorem 2 hold. Then*

$$\begin{aligned} R_{0,n}^{na}(D) &= \max_{s \leq 0} \max_{\lambda \in \Psi_s} \left\{ sD(n+1) + \sum_{i=0}^n \int \log(\lambda_i(x^i, y^{i-1})) \right. \\ &\quad \left. \times P_{0,i-1}(dx^{i-1}, dy^{i-1}) \otimes p_i(dx_i; x^{i-1}) \right\} \end{aligned}$$

where $\Psi_s \triangleq \{\lambda \triangleq \{\lambda_i(x^i, y^{i-1}) \geq 0 : i = 0, 1, \dots, n\} : \int e^{s\rho(x_i, y_i)} \lambda_i(x^i, y^{i-1}) P_{0,i}(dx^i|y^{i-1}) \leq 1, i = 0, 1, \dots, n\}$.

Proof: The derivation is found in [12, App. E]. ■

IV. APPLICATIONS VIA EXAMPLES

In this section, we describe some applications of information NRDF using the following two working examples: (i) the BSMS(p), (ii) the multidimensional Gaussian stationary source.

Bound and RL due to Causal Codes. Let $R(D)$ denotes the OPTA by noncausal codes [3], and $r^c(D)$ the OPTA by causal codes [13]. Then we have the following bounds.

$$R(D) \stackrel{(b)}{\leq} R^{na}(D) \stackrel{(c)}{\leq} r^c(D) \quad (10)$$

where (b) follows from the fact that $R(D)$ is optimized over a larger set than that of $R^{na}(D)$, and (c) follows by the converse coding theorem and [13]. Since the OPTA by noncausal codes for sources with memory is often unknown (unless one consider memoryless or Gaussian sources), then $R^{na}(D)$ can be used to find an upper bound to the OPTA by noncausal codes. For memoryless sources $R(D) = R^{na}(D)$, and this bound is tight. Moreover, since $r^c(D) - R(D) \geq R^{na}(D) - R(D)$, we can find the RL of causal codes with respect to the noncausal codes using $R^{na}(D)$.

Noisy Coding Theorem (Source-Channel Matching). An operational definition for $R^{na}(D)$ can be established by using symbol-by-symbol transmission, provided for a given source and distortion function we can find the optimal reproduction distribution, and then realize it over an encoder-channel-decoder, so that the source is matched to the channel. We give an example for multidimensional Gaussian stationary sources providing a noisy coding theorem for $R^{na}(D)$.

A. BSMS(p): Exact Solution, Bounds, and Rate Loss

Consider a BSMS(p), with stationary transition probabilities $\{P_{X_i|X_{i-1}}(x_i|x_{i-1}) : (x_i, x_{i-1}) \in \{0, 1\} \times \{0, 1\}\}$ given by $P_{X_i|X_{i-1}}(0|0) = P_{X_i|X_{i-1}}(1|1) = 1 - p$, $P_{X_i|X_{i-1}}(1|0) = P_{X_i|X_{i-1}}(0|1) = p$, $i \in 0, 1, \dots$, and single letter Hamming distortion criterion, $\rho(x, y) = 0$ if $x = y$ and $\rho(x, y) = 1$ if $x \neq y$. The solution to the NRDF is given to the next theorem.

Theorem 4. *For a BSMS(p) and single letter Hamming distortion*

$$R^{na}(D) = \begin{cases} H(m) - H(D) & \text{if } D \leq \frac{1}{2} \\ 0 & \text{otherwise} \end{cases} \quad (11)$$

where $m = 1 - p - D + 2pD$, and the optimal (stationary) reproduction distribution is

$$P_{Y_i|X_i, Y_{i-1}}^*(y_i|x_i, y_{i-1}) = \begin{bmatrix} \alpha & \beta & 1-\beta & 1-\alpha \\ 1-\alpha & 1-\beta & \beta & \alpha \end{bmatrix}$$

where $\alpha = \frac{(1-p)(1-D)}{1-p-D+2pD}$, $\beta = \frac{p(1-D)}{p+D-2pD}$.

Proof: The proof is found in [12, Th. IV.11]. ■

Note that for $p = \frac{1}{2}$, then BSMS($\frac{1}{2}$) is the IID Bernoulli source, and $R^{na}(D) = 1 - H(D) \equiv R(D)$, $D < \frac{1}{2}$, as expected. The graph of $R^{na}(D)$ is illustrated in Fig. 1.

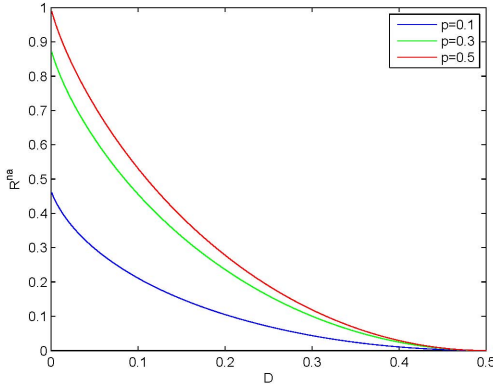


Fig. 1. $R^{na}(D)$ for different values of parameter p .

Bounds on $R(D)$. The classical RDF for the BSMS(p) is only known for the distortion region $0 \leq D \leq D_c$ [5], while for the rest distortion region only bounds are known [6]. Fig. 2 shows the graph of $R(D)$ for $0 \leq D \leq D_c$, Berger's lower and upper bounds [6], SLB, and the upper bound based on $R^{na}(D)$. We observe that for $p = 0.25$, the upper bound based on $R^{na}(D)$ does slightly better than Berger's upper bound. However, for small values of D , we have observed that Berger's upper bound

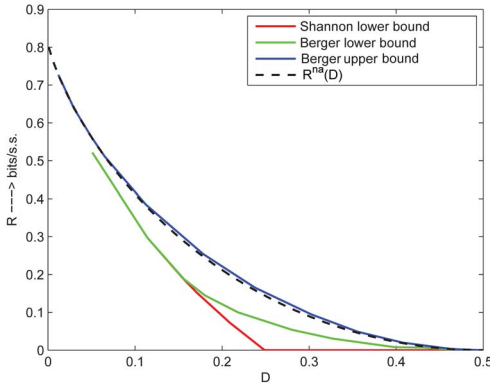


Fig. 2. $R(D)$ for BSMS(p) for $0 \leq D \leq D_c$ and Bounds for $p = 0.25$.

fails to be tight, while the one based on $R^{na}(D)$ is tight [12,

Sec. V.C].

RL of Causal Codes. By utilizing the bound $R^{na}(D) \geq R(D)$, we can deduce that the RL of causal codes for the BSMS(p) cannot exceed $R^{na}(D) - R(D) = H(m) - H(q)$, $0 \leq D \leq D_c$, where $R(D) = H(q) - H(D)$, $p = 1 - q$, $q \leq \frac{1}{2}$, $D \leq D_c = \frac{1}{2} \left(1 - \sqrt{1 - \left(\frac{q}{p} \right)^2} \right)$. Note that the exact value of RL is only given for the region $0 \leq D \leq D_c$, where the exact solution of $R(D)$ is known. Beyond this region, upper and lower bounds for RL can be found [12, Sec. V.C].

B. Multidimensional Gaussian Stationary Sources: Source-Channel Matching and Rate Loss

In this section, we consider a vector partially observable Gaussian-Markov process and we compute explicitly the closed form expression of $R^{na}(D)$. This expression makes feasible the matching of the source to the channel.

Consider the following multidimensional partially observed linear Gauss-Markov system

$$\begin{cases} Z_{t+1} = AZ_t + BW_t, & Z_0 = z, \quad t \in \mathbb{N} \\ X_t = CZ_t + NV_t, & t \in \mathbb{N} \end{cases} \quad (12)$$

where $Z_t \in \mathbb{R}^m$ is the state (unobserved) process and $X_t \in \mathbb{R}^p$ is the information source, obtained from noisy measurements of CZ_t . In this application the objective is to compress the sensor data, which is the only observable information. Next, we introduce certain assumptions which are standard in infinite horizon Kalman Filter [14], and they are also sufficient for existence of the limit, $\lim_{n \rightarrow \infty} \frac{1}{n+1} R_{0,n}^{na}(D)$.

(E1) (C, A) is detectable and $(A, \sqrt{BB^T})$ is stabilizable, $(N \neq 0)$; **(E2)** the state and observation noise $\{(W_t, V_t) : t \in \mathbb{N}\}$ are Gaussian IID vectors $W_t \in \mathbb{R}^k$, $V_t \in \mathbb{R}^d$, mutually independent with parameters $N(0, I_{k \times k})$ and $N(0, I_{d \times d})$, independent of the Gaussian RV Z_0 , with parameters $N(\bar{z}_0, \bar{\Sigma}_0)$; **(E3)** the distortion function is single letter defined by $d_{0,n}(x^n, y^n) \triangleq \sum_{t=0}^n \|x_t - y_t\|_{\mathbb{R}^p}^2$.

According to Theorem 2, the optimal stationary reproduction distribution is given for $s \leq 0$ by

$$P_{Y_t|Y^{t-1}, X_t}^*(dy_t|y^{t-1}, x_t) = \frac{e^{s\|y_t - x_t\|_{\mathbb{R}^p}^2} P_{Y_t|Y^{t-1}}^*(dy_t|y^{t-1})}{\int_{\mathcal{Y}_t} e^{s\|y_t - x_t\|_{\mathbb{R}^p}^2} P_{Y_t|Y^{t-1}}^*(dy_t|y^{t-1})}. \quad (13)$$

Note that the exponential quadratic term in (13) implies that $P_{Y_t|Y^{t-1}, X_t}(\cdot|y^{t-1}, x_t)$ is conditionally Gaussian (using completion of squares if necessary). Hence, the channel connecting $\{X_t : t \in \mathbb{N}\}$ to $\{Y_t : t \in \mathbb{N}\}$ has the general form

$$Y_t = \bar{A}X_t + \bar{B}Y^{t-1} + V_t^c, \quad t \in \mathbb{N} \quad (14)$$

where $\bar{A} \in \mathbb{R}^{p \times p}$, $\bar{B} \in \mathbb{R}^{p \times tp}$, and $\{V_t^c : t \in \mathbb{N}\}$ is an independent sequence of Gaussian vectors with zero mean and covariance $\text{cov}(V_t^c) = Q = \text{diag}\{q_1, \dots, q_p\}$. Consider a pre-encoder introducing the Gaussian error process $\{K_t : t \in \mathbb{N}\}$, $K_t \triangleq X_t - \mathbb{E}\{X_t|Y^{t-1}\}$ and its steady state covariance Λ_∞ ,

$\Lambda_\infty = \lim_{n \rightarrow \infty} \Lambda_t$, $\Lambda_t \triangleq \mathbb{E}\{K_t K_t^{tr}\}$, $t \in \mathbb{N}$. Let E_∞ be a unitary matrix such that

$$E_\infty \Lambda_\infty E_\infty^{tr} = \text{diag}\{\lambda_{\infty,1}, \dots, \lambda_{\infty,p}\}, \quad \Gamma_t \triangleq E_\infty K_t, \quad t \in \mathbb{N}. \quad (15)$$

Analogously, introduce the process $\{\tilde{K}_t : t \in \mathbb{N}\}$ defined by $\tilde{K}_t \triangleq Y_t - \mathbb{E}\{X_t|Y^{t-1}\} \equiv Y_t - \hat{X}_{t|t-1}$, $\tilde{\Gamma}_t = E_\infty \tilde{K}_t$. It is easily shown that $d_{0,n}(X^n, Y^n) = d_{0,n}(K^n, \tilde{K}^n) = \sum_{t=0}^n \|\tilde{K}_t - K_t\|_{\mathbb{R}^p}^2 = \sum_{t=0}^n \|\tilde{\Gamma}_t - \Gamma_t\|_{\mathbb{R}^p}^2$. Using basic properties of conditional entropy we

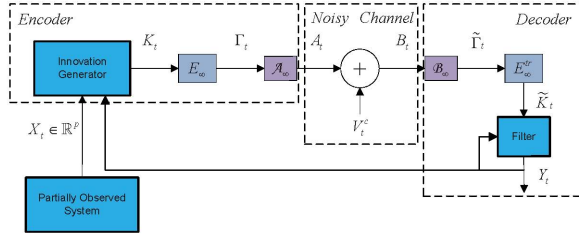


Fig. 3. Realization of the optimal stationary reproduction distribution.

can show that $R^{na}(D) = \lim_{n \rightarrow \infty} \frac{1}{n+1} R_{0,n}^{na,K^n,\tilde{K}^n}(D) = \lim_{n \rightarrow \infty} \frac{1}{n+1} R_{0,n}^{na,\Gamma^n,\tilde{\Gamma}^n}(D)$. Next, we state the main result.

Theorem 5. Under Assumptions (E1)-(E3), the information NRDF rate for (12) is given by

$$R^{na}(D) = \frac{1}{2} \sum_{i=1}^p \log \left(\frac{\lambda_{\infty,i}}{\delta_{\infty,i}} \right)$$

where $\text{diag}\{\lambda_{\infty,1}, \dots, \lambda_{\infty,p}\} = \lim_{t \rightarrow \infty} E_t \Lambda_t E_t^{tr} = E_\infty \Lambda_\infty E_\infty^{tr}$, $\Lambda_\infty = C \Sigma_\infty C^{tr} + N N^{tr}$

$$\delta_{\infty,i} \triangleq \begin{cases} \xi_\infty & \text{if } \xi_\infty \leq \lambda_{\infty,i} \\ \lambda_{\infty,i} & \text{if } \xi_\infty > \lambda_{\infty,i} \end{cases}, \quad i = 2, \dots, p$$

and ξ_∞ is chosen such that $\sum_{i=1}^p \delta_{\infty,i} = D$. Define $H_\infty = \lim_{t \rightarrow \infty} H_t$, $H_t \triangleq \text{diag}\{\eta_{t,1}, \dots, \eta_{t,p}\}$, $\eta_{t,i} = 1 - \frac{\delta_{t,i}}{\lambda_{t,i}}$, $i = 1, \dots, p$, $\mathcal{B}_\infty = \lim_{t \rightarrow \infty} \mathcal{B}_t = \sqrt{H_\infty \Delta_\infty Q^{-1}}$, $\mathcal{B}_t \triangleq \sqrt{H_t \Delta_t Q^{-1}}$, $\Delta_\infty = \lim_{t \rightarrow \infty} \Delta_t$, $\Delta_t = \text{diag}\{\delta_{t,1}, \dots, \delta_{t,p}\}$, $t \in \mathbb{N}$. Moreover, Σ_∞ is the steady state covariance of the error $Z_t - \mathbb{E}\{Z_t|Y^{t-1}\} \sim N(0, \Sigma_\infty)$ of the Kalman filter given by

$$\begin{aligned} \hat{Z}_{t+1|t} &= A \hat{Z}_{t|t-1} \\ &+ A \Sigma_\infty (E_\infty^{tr} H_\infty E_\infty C)^{tr} M_\infty^{-1} (Y_t - C \hat{Z}_{t|t-1}) \\ \Sigma_\infty &= A \Sigma_\infty A^{tr} \\ &- A \Sigma_\infty (E_\infty^{tr} H_\infty E_\infty C)^{tr} M_\infty^{-1} (E_\infty^{tr} H_\infty E_\infty C) \Sigma_\infty A^{tr} + B B^{tr} \\ M_\infty &= E_\infty^{tr} H_\infty E_\infty C \Sigma_\infty (E_\infty^{tr} H_\infty E_\infty C)^{tr} \\ &+ E_\infty^{tr} H_\infty E_\infty N N^{tr} (E_\infty^{tr} H_\infty E_\infty)^{tr} + E_\infty^{tr} \mathcal{B}_\infty Q \mathcal{B}_\infty^{tr} E_\infty \end{aligned}$$

where $\hat{Z}_{t|t-1} \triangleq \mathbb{E}\{Z_t|Y^{t-1}\}$ and $\hat{Z}_0 = \mathbb{E}\{Z_0|Y^{-1}\}$, $Z_0 \sim N(0, \Sigma_\infty)$.

Proof: The proof is found in [12, App. F]. ■

Source-Channel Matching. In view of Fig. 3, the conditional distribution of NRDF is realized via an encoder-channel-decoder. Moreover, the channel consists of parallel additive Gaussian noisy channels with feedback defined by

$$B_{t,i} = A_{t,i}(X_t, B^{t-1}) + V_{t,i}^c, \quad t \in \mathbb{N}, \quad i = 1, \dots, p.$$

Recall that the capacity of a parallel memoryless Gaussian channel with feedback subject to a power constraint $\frac{1}{n+1} \mathbb{E}\{\sum_{t=0}^n \|A_t\|_{\mathbb{R}^p}^2\} \leq P$, is given by $C(P) = \lim_{n \rightarrow \infty} \frac{1}{n+1} \sum_{t=0}^n \sum_{i=1}^p \log |1 + \mathbb{E}\{(A_{t,i})^2\} Q^{-1}| = \frac{1}{2} \sum_{i=1}^p \log(1 + \frac{P_{\infty,i}}{q_i})$, $\sum_{i=0}^p P_{\infty,i} = P$, $P_{\infty,i} = \lim_{n \rightarrow \infty} E\{(A_{t,i})^2\}$. As a result, for a given $D \geq 0$, we can let $P = D$, i.e., $\frac{P_{\infty,i}}{q_i} = \frac{\lambda_{\infty,i}}{\delta_{\infty,i}} - 1$, then $C(P) = R^{na}(D)$, and the end-to-end distortion is satisfied.

RL of Zero-Delay Codes. The source distribution $\{X_t : t \in \mathbb{N}\}$ in (12) is Gaussian, hence we can compute the OPTA by noncausal codes, $R(D)$, by using power spectral density expression [3]. The RL of causal and zero-delay codes with respect to the noncausal codes is precisely $\frac{1}{2} \sum_{i=1}^p \log \left(\frac{\lambda_{\infty,i}}{\delta_{\infty,i}} \right) - R(D)$ bits/sample.

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